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Abstract

Supermajorities have their advantage as well as their disadvantage: they provide an hedge against being in the minority, but they make being in the majority less likely. We characterize this trade-off and compute the most preferred majority threshold. The relevant parameters are voting power, risk aversion and pessimism. People who feel powerful prefer low thresholds. High thresholds are preferred by risk averters or by those who are pessimistic about being in the majority.

Further we study constitutional agreements on the voting rule. Members of the constituent assembly are heterogeneous in the parameters above. We show that weak and minority members succeed in pushing forward on high and protective rules.

Keywords: majority rule, supermajority, risk aversion, weighted votes, constitutions, tyranny of the majority.

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1 Introduction

Voting is probably the most common way to make collective decisions, not only in legislatures, but also in international committees, corporations, associations, and even in condominiums. Majorities sometimes take decisions that make some individuals worse off than the status quo. In this case we say that those individuals are expropriated or tyrannized by the majority. The probability of being tyrannized is low when voting rules are more protective. In this case, however, the chance of forming a favorable majority is low too. The voting rules such as the majority threshold can be viewed as the outcome of the trade-off between protectiveness and decisiveness. In this paper we study preferences for majority thresholds and their role in constitutional negotiations.

We treat voting as a lottery in which the probability of winning or losing depends on the majority threshold. The stylized situation is a legislature in which two reform proposals are put forward for voting. A voter “wins” if her most preferred policy passes. She “loses” if the other one passes, as we assume that her payoff is worse than the status quo. The outcome is uncertain because she does not know how the others will vote. In this lottery probabilities depend on the required majority threshold. If the threshold is low, making decisions is easy. In this case, however, not only winning is rather likely, but also losing. There is a trade-off: decisiveness versus protectiveness. Under simple conditions, a voter has one single most preferred majority threshold. We characterize it, and we show that a voter wants a low threshold (i.e. high decisiveness and low protectiveness) when she is optimistic about how the others will vote, or she does not dislike risk, or when she commands a lot of votes. Interestingly, the most preferred threshold in many cases is either the bare majority or unanimity.

The underlying idea is that the demand for decisiveness comes from powerful agents (say powerful parties, large political factions, or big states in federal contexts), from those who are optimistic because they think that their preferences are similar to the mass (say, citizens who feel themselves

in the majority), from those who get a lot from winning or those who do not strongly avert the risk of losing. By contrast, demand for protection comes from risk averters or weak minorities with high potential losses from tyranny or expropriation.

We should make clear from the outset that there is no bargaining at the “legislative” stage in this model: the two competing proposals are exogenous. An interpretation is that two parties make divergent proposals which possibly reflect the broadest consensus of their electoral base, and no side-payments are possible in order to collect a wider support. Thus the society is split in two distinct groups or classes, and each proposal can benefit only one or the other. Individuals do not know the exact proportions of the two groups. In this context we are able to endogenize the threshold in a highly stylized and simplified way. We will come back to this point in the Conclusion.

We assume that fixing voting rules for any future decisions within a constitution is more efficient than bargaining on every single decision in the future. Thus we use individual preferences for majority thresholds to build a constitutional stage.¹ The members of the constituent assembly ignore which decisions will be made in the future but they know their degree of risk aversion. They can also know whether they belong to a minority with specific interests or not, and whether they will have high or low influence on future voting. The idea is that weak members, risk averters or minorities feel more subject to expropriation risk in future decisions. We show that this gives them consistent leverage in demanding for higher supermajorities. Our claim is that negotiations amongst “unequals” are likely to yield quite protective constitutions.

¹Constitutional negotiations are difficult to model under a unique perspective. They often come after extended periods of political instability or after revolutions. They may involve radically different situations or actors. In order to give general validity to our analysis, we avoid the complications or the arbitrariness of extensive-form specifications. We opt for a cooperative and axiomatically founded approach, the Nash bargaining. But we expect our findings to survive in many alternative strategic models.

In Section 3 we give other arguments in favor of our modelling choice.

In reality the simple majority is less frequent than one may think. Voting rules in institutions and committees are often far from being decisive; they are rather protective. For example, most governments adopt bicameral systems which are de facto a form of supermajority: it is not easy for a future majority to repeal acts passed by a previous two-house majority. Specific supermajorities are required in cases where a broad consensus rather than a bare majority is sought-after. For instance, the U.S. Federal Constitution requires a two-thirds vote to override a presidential veto, to ratify a treaty, or to expel a member of Congress. Three fifths of the full Senate must approve any waiver regarding balanced budget provisions or points of order concerning legislation that would violate a budget approved by Congress. Recently the Lisbon Treaty has adopted a double supermajority for the EU Council. In international treaties, members can exercise vetoes when decisions concern their crucial interests (the Council of the EU, the UN Security Council). When corporate boards vote on major actions (mergers and acquisitions, major capital expansions, etc.) usually high thresholds are required.

In the U.S., there is an ongoing debate about the proposed ‘Tax Limitation Constitutional Amendment’ that would require a two thirds majority in the House and Senate anytime a decision leads to a federal debt or tax increase. Fifteen states, comprising one third of the U.S. population, already have supermajority requirements for state tax increases. In legislatures where minorities are represented, small members’ weights are proportionally higher or supermajority thresholds occur more frequently. During the recent debate on the “EU Constitution”, small member countries have opposed majority threshold reductions in order to keep their blocking power for future decisions. These facts are consistent with the model presented in this paper.

Traditionally the issue of supermajorities has been posed as a way to mitigate the “tyranny of the majority” (Buchanan and Tullock, 1962).² Hayek (1960) argues that the simple majority

²A noticeable literature exists on several aspects, other than the ones considered here, regarding the constitutional

rule does not have any superior appeal, since it may lead to excessive costs in terms of individual freedom. Rae (1969) is the first who modelled the relationship between voting rules and uncertainty. Uncertainty concerns the effects of a law. Losses and gains are equal and equally likely. Agents are risk neutral. Rae suggests (and Taylor, 1969, formally proves) that the simple majority is the only rule that minimizes the expected cost of being part of the minority. The present paper considers a wider range of situations (asymmetric gains and losses, weighted votes, different degrees of risk aversion and confidence). Thus Rae's result emerges as a special case of our model.³

Weighted voting creates variations in the likelihood of winning and thus variations in the preferences about the voting rule. Specifically, those with higher weight will tend to prefer a low supermajority. This is certainly an old idea, but, to the best of our knowledge, it has not been fully developed theoretically.⁴

choice of voting rules. Diermeier and Myerson (1999) offer institutional arguments for supermajorities: two competing legislative chambers introduce supermajorities or internal vetoes so as to assert more power vis-à-vis the other chamber. Krehbiel (1998) suggests that supermajority rules are used to preserve current policy choices into the future or to give more legislative power to the president rather than the congress. Weingast (1979) and other scholars on universalism suggest that, since small majority coalitions are prone to cycles, there may be efficiency reasons for the use of norms of inclusive, consensual decision-making; i.e. supermajority rules. In fact, Caplin and Nalebuff (1988) show that a 64% threshold prevents cyclicity with a small range of restrictions on agents' preferences. For an excellent overview of pros and cons of supermajority rules, with detailed references, see McGann (2006).

Finally, Feddersen and Pesendorfer (1998) and Austen-Smith and Feddersen (2006) analyze how voting rules affect information aggregation in voting context with private information.

³The special case is the following: same (negligible) voting power for all voters, gains equal to losses, total unawareness of how the others will vote. The reader may find a direct relation of these conditions with the anonymity and neutrality axioms of the May's theorem on simple majority (May, 1952). More precisely, same voting power satisfies anonymity; symmetry in gains and losses together with unawareness guarantees neutrality. For discussion and refinements of the May's theorem, see Dasgupta and Maskin (2008).

⁴Cooperative and non-cooperative models of legislative bargaining have explored the impact of voting weights on the division of the spoils, but not the relationship between weights and the preferred threshold. Cooperative

Here we try to answer the following question: “Which voting rule will a group adopt?” Recent papers have addressed the same “constitutional” question. Aghion and Bolton (2002) show that the optimal majority threshold increases in the expected cost of compensating the losing minority, when agents do not know ex-ante if they will lose or gain from the provision of a public good. Messner and Polborn (2004) suggest that old people are more conservative than young people because they pay more taxes or get benefits from reforms for a shorter time. Thus they prefer higher thresholds. Barberà and Jackson (2006) endogenize weighted votes and study their role in an indirect democracy. They characterize the efficient voting rule, that should be a mixture of weights and supermajority. Holden (2009) determines the majority rule with a spatial approach in which voters know the distribution of bliss points. Some of his results are similar to ours in Section 2. However he does not consider voting power and, since agents are behind a veil of ignorance, they all agree ex-ante on the same voting rule. Aghion, Alesina and Trebbi (2004) analyze the constitutional choice about the level of insulation of political leaders. The optimal degree of insulation depends on the cost of compensating the losers, the social benefits of policy reforms, the uncertainty about gains and losses, the degree of risk aversion. These models are close to ours in many respects. However there are substantial differences. First, we do not consider social efficiency; our analysis is strictly positive. Second, some of these models consider a representative

analysis predicts a non-monotone relationship between weight and bargaining power (for general analysis with many references therein see Owen, 1995; Felsenthal and Machover, 1998; Benoit and Korhauser, 2002). The reader will find a relationship between Banzhaf’s (1965) power index and our model. Snyder, Ting, and Ansolabehere (2005) found that in a non-cooperative bargaining game weights and spoils are proportional. More on this below.

Another branch of the literature has focused on the rationale of voting weights (e.g. Penrose, 1946; Nitzan and Paroush, 1982; Shapley and Grofman, 1984).

Barberà and Jackson (2006) approach weights from a normative viewpoint: weights are a tool for taking intensity of preferences into account in the collective choice.

agent with no constitutional negotiation, others describe the constitutional stage as a voting game.⁵ We present the constitutional stage as a Nash *unanimous* bargaining game amongst *heterogeneous* members of a constituent assembly. Third, in these models members have the same voting weight (one vote each); we also consider asymmetries in voting power. Fourth, in these models agents share ex-ante the same degree of uncertainty about policy outcomes; in our model, assembly members can have different subjective expectations. Finally, since we use generic utility functions we perhaps provide a more general treatment of risk aversion.

Existing literature has studied optimism (Buchanan and Faith, 1980, 1981; Zorn and Martin, 1986) and risk aversion (among many others, Harrington, 1990; Aragues and Postlewaite, 2002; Berinsky and Lewis, 2007). However, differently from us, this literature has looked at the calculus of voting, rather than the choice of voting rules.

In the present paper individuals evaluate the chance of being in the majority against the probability of falling into the minority. This exercise of weighting future positions with their probabilities connects this paper to Harsanyi's (1955) approach to the social welfare function. The relationship between this approach and the constitutional choice of voting rules has been stressed by Mueller (1973).

The rest of the paper is structured as follows. Section 2 presents the setup of the legislative stage. This is the first block of our theory of constitutional choice. Here we compute the optimal majority threshold and we analyze how it depends on individual features (degree of optimism, risk

⁵In the latter case, some conceptual questions remain eluded. For example, "Why and when have agents agreed to vote on a constitution? How have they determined the threshold for voting on the voting threshold?" Basically we think that such questions remain unanswered if one does not assume that at some stage of a non-coercive constitutional process there has been some unanimous consent. More on this in Section 3.

A different question is requiring *within* a constitution that future constitutional amendments can be made by voting.

attitudes and voting weights). A simplified discrete version of the legislative model is provided in Appendix 4.2. This version is more suitable for a committee with a small number of actors. In Section 3 we present the second block: a constitutional stage in which we compute the equilibrium threshold. Section 4 concludes. All the proofs are in Appendix 4.1.

2 The Legislative lottery

Consider a set $N = \{1, \dots, n\}$ of agents who make common decisions by voting. Let $\{q; w_1, \dots, w_n\}$ be the set of voting rules, where q is the majority threshold and w_i represents agent i 's ($i = 1, \dots, n$) number of votes. Denote by m the sum of votes, $\sum_N w_i$, and assume that $q > \frac{m}{2}$. Suppose that the assembly has to deliberate on two exogenous policy proposals, α and β , and assume that N is partitioned in two subsets, the α -types and the β -types. If the adopted policy is α then all the α -types gain with respect to the status quo and the β -types lose. If β gets adopted, the distribution of gains and losses is reversed. If no policy passes, then the status quo, ς , remains. We assume that whenever a majority coalition has the chance to be formed on a policy, either α or β , that policy is put forward. Abstention is not possible. Thus, all the α -types vote in favor of α and all the β -types vote against it, and vice versa. Equivalently, a α -type ranks policy α first, the status quo second, and policy β third. As for a β -type, the ordering is reversed.⁶

With a slight variation in the meaning of variables, this “legislative” framework applies to electoral competitions too. In this case, α and β represent the *exogenous* electoral platforms proposed by two candidates to lead the executive branch of a government. Say, an α -type citizen

⁶Probably, the simplest way to look at α and β is considering them as purely redistributive policies: α is a tax levied on β -types and totally transeferred to α -types (and vice versa). However, many kinds of policy alternatives, without necessarily redistributive contents, but possibly with social or ideological aspects involved, can be described in the same way.

has an incentive to reduce the ability of the β -candidate to pass platform β . This can be done by setting up a high threshold for the parliamentary approval. In Aghion, Alesina and Trebbi's (2004) terminology this would mean that any future leader, either β - or α -candidate, will be less "insulated".

Consider now an agent j . Let us describe her prospect from voting. Suppose she is an α -type. We say that she "wins" in case policy α passes; she "loses" if β passes. In the latter case, we say also that she suffers the tyranny of an undesired majority. Thus, "winning" is better than the status quo, and the status quo is better than "losing". Let $\{\alpha, \varsigma, \beta\}$ be the policy space, and let $u_j : \{\alpha, \varsigma, \beta\} \rightarrow \mathbb{R}$ be agent j 's utility function:⁷

$$u_j(\alpha) > u_j(\varsigma) > u_j(\beta).$$

We are interested in calculating j 's subjective probabilities of winning or losing. She knows her type, but assume that she is uncertain about the type of the other agents. She subjectively thinks that any other agent in $N \setminus j$ is of type α with probability p .⁸ Thus p is her subjective probability that any other agent will vote for α . Conversely, $(1 - p)$ is the probability that any other agent votes for β . We say that p represents agent j 's degree of optimism about how the others will vote: j is optimistic when she thinks that the others are quite likely to share her own preferences. Thus p captures how an agent feels to be different/similar to the others. For example, members of ethnic minorities are likely to have a low p . This parameter may also reflect individual psychological traits.⁹

⁷We can apply differential calculus to u_j if α , ς and β are real numbers. If we let them to be any kind of policy (such as behavioral norms, alternative foreign policies, technical standards,...), then one may assume that there exists a function $g_j(.) : \{\alpha, \varsigma, \beta\} \rightarrow \mathbb{R}$ that identifies the monetary consequence of the policy for agent j . In this case, u_j is an increasing function defined in the codomain, \mathbb{R} , of g_j .

We explicitly consider monetary outcomes in Lemma 2, and we introduce g_j in its proof (see Appendix 4.1).

⁸Notice that, in order to save on notation, p has not been indexed by j .

⁹Instead of thinking of p as an exogenous parameter, one can alternatively interpret it as a Bayesian updating of

Call $S_\alpha \subseteq N \setminus j$ the coalition of the “other agents” who vote for policy α . Agent j ’s probability of winning, $\Pr\{\alpha\}$, is given by the probability that S_α gets at least $q - w_j$ votes. Then agent j “adds” her own w_j votes and the majority forms. Given j ’s uncertainty regarding the voting behavior of other agents, the sum of votes in S_α is a random event that behaves as the sum of $n - 1$ independent random variables, Z_i , ($i = 1, \dots, n; i \neq j$); where $Z_i = w_i$ with probability p , and $Z_i = 0$ with probability $(1 - p)$. Suppose the number of agents is sufficiently large and let us apply the Central Limit Theorem. Thus, the sum of votes collected by S_α is distributed like a normal with parameters $\mu_\alpha = \sum_{i \in N \setminus j} w_i p$, and $\sigma_\alpha^2 = \sum_{i \in N \setminus j} w_i^2 p(1 - p)$. Let $f^\alpha(\cdot)$ be its density function.

With a few number of voters (say less than twenty) the approximation of the Central Limit Theorem becomes quite large. Thus a discrete model with exact probability distributions is more appropriate. We present a discrete model in Appendix 4.2. We show that the main findings of this Section continue to hold, but in order to draw general results we need the population to be sufficiently large. With the discrete model we lose the benefits of differential calculus, thus the math is much heavier. Moreover, in a discrete model there is no scope for weighted votes, since no exact distribution functions can be built. In addition, with discrete functions the constitutional analysis of Section 3 is virtually impossible. These limitations of the approach with discrete distributions persuaded us to opt for a continuous model.

j ’s subjective probability of winning is given by the probability that the sum of “favorable” votes lies in $[q - w_j, m - w_j]$:

$$\Pr\{\alpha\} = \int_{q-w_j}^{m-w_j} f^\alpha(x) dx. \quad (1)$$

Conversely, j ’s subjective probability of losing is the probability of forming a majority coalition priors, based on informative signals, such as in exit polls. Good pools induce optimism.

on policy β ; i.e. the probability that the sum of “unfavorable” votes is in $[q, m - w_j]$:

$$\Pr\{\beta\} = \int_q^{m-w_j} f^\beta(x) dx \quad (2)$$

where $f^\beta(\cdot)$ is a normal density function with parameters: $\mu_\beta = \sum_{N \setminus j} w_i(1 - p)$, and $\sigma_\beta^2 = \sigma_\alpha^2 = \sigma^2 = \sum_{N \setminus j} w_i^2 p(1 - p)$.

Finally, j ’s subjective probability that the status quo prevails is the probability that no majority coalition is formed:

$$\Pr\{\varsigma\} = 1 - \Pr\{\alpha\} - \Pr\{\beta\}. \quad (3)$$

Of course, with the simple majority, $\Pr\{\varsigma\}$ is always zero.¹⁰ With unanimity, if p is not “too close” to one or to zero, the status quo is “almost” certain.¹¹

Thus, j ’s voting prospect can be described as a lottery,

$$L_j = (\alpha, \Pr\{\alpha\}; \beta, \Pr\{\beta\}; \varsigma, \Pr\{\varsigma\}),$$

with three possible outcomes and attached subjective probabilities as defined in (1-3).¹²

Agent j ’s expected utility from the voting lottery is

$$EU_j(L_j(q)) = \Pr\{\alpha\} \cdot u_j(\alpha) + \Pr\{\beta\} \cdot u_j(\beta) + \Pr\{\varsigma\} \cdot u_j(\varsigma) \quad (4)$$

¹⁰When m is odd, this is trivial. When m is even, we assume that a possible (even though unlikely) tie is solved by flipping a fair coin.

¹¹Recall that the normal is an acceptable approximation of the true distribution when the number of agents is sufficiently high. For example, with 40 unweighted agents who vote for α with probability $p = 0.9$, the probability of reaching unanimity on policy α is less than 1.5%, whereas the status quo probability is more than 98.5%.

¹²A natural interpretation of this lottery is that in the legislative stage alternative α is posed against alternative β . This might sound strange if one is used to thinking of the legislative process as a pairwise competition between the current status quo and any proposal. However, no substantial change would occur in the voting prospect if one assumes that, in a first round, any of the two alternatives (say α) is posed against the status quo ς ; the winning one becomes the new status quo; then, in a second round, the other alternative, β , is posed against the new status quo.

Observe that subjective probabilities (1-3) depend on the majority threshold q , voting weights w_i , and agent j 's degree of optimism p . Also, j 's degree of risk aversion affects outcome utilities in (4).

Specifically, changes in q affect L_j and, as a consequence, influence j 's expected utility. Is there any “preferred” threshold q_j^* that yields a voting lottery with the highest expected utility? In Section 2.1, we show that q_j^* exists and is unique.¹³ Further, we explore how the optimal threshold depends on optimism, power and risk aversion.

2.1 Optimal thresholds

In this Section we compute the threshold q_j^* that maximizes agent j 's expected utility in (4). The first-order condition (FOC) for a stationary point, q_j^0 , is $\left. \frac{\partial EU_j(L_j)}{\partial q} \right|_{q_j^0} = 0$. Thus,

$$f^\alpha(q_j^0 - w_j) \cdot [u_j(\alpha) - u_j(\varsigma)] = f^\beta(q_j^0) \cdot [u_j(\varsigma) - u_j(\beta)]. \quad (5)$$

From (5) it is clear that the agent balances the negative marginal impact of the threshold on reducing the expected benefit of belonging to the majority (left-hand side) with the positive marginal impact of the threshold on reducing the expected loss of falling into the minority (right-hand side). Recall that f^α and f^β are two normal densities, thus we have an analytical expression for them. It is easy to see that the stationary point that satisfies the FOC is unique and it is given by:

$$q_j^0 = \frac{m}{2} + \frac{\sigma^2 \ln RASQ_j}{w_j + \mu_\alpha - \mu_\beta} \quad (6)$$

where

$$RASQ_j = \frac{u_j(\varsigma) - u_j(\beta)}{u_j(\alpha) - u_j(\varsigma)}. \quad (7)$$

¹³Notice that, if $u_j(\alpha) \geq u_j(\beta) \geq u_j(\varsigma)$, the problem of finding the optimal threshold becomes trivial: agent j always (at least weakly) prefers the bare majority rule.

$RASQ_j$ in (7) is the Relative Advantage of the Status Quo. It is the ratio between j 's benefits of not being tyrannized by an unfavorable majority, $u_j(\varsigma) - u_j(\beta)$, and the benefits of being part of a favorable majority that “tyrannizes others”, $u_j(\alpha) - u_j(\varsigma)$. High values of $RASQ_j$ reveal high sensitivity to being tyrannized, rather than to winning. Let us see why q_j^0 and $RASQ_j$ are positively related in (6). Assume that $RASQ_j$ is high, and suppose that, starting from a low value, q increases. The chance to form a majority is lower. Thus both winning and losing are less likely. Is j better off or not? Since she is highly sensitive to losing, her expected utility raises. In general, agents with high $RASQ$ tend to prefer high q . We will prove this result in the next Section.

In order to determine the most preferred threshold, q_j^* , recall that, by assumption, $q_j^* \in [q^s, m]$, where q^s is the simple majority threshold.¹⁴ Also we have to study the sign of the second-order condition (SOC) in the stationary point, $\left. \frac{\partial^2 EU_j(L_j)}{\partial q^2} \right|_{q_j^0}$, that is:

$$-f_q^{\alpha'}(q_j^0 - w_j) \cdot [u_j(\alpha) - u_j(\varsigma)] + f_q^{\beta'}(q_j^0) \cdot [u_j(\varsigma) - u_j(\beta)] \quad (8)$$

with $f_q^{\alpha'}(q - w_j) = \frac{\partial f^{\alpha}(q - w_j)}{\partial q}$ and $f_q^{\beta'}(q) = \frac{\partial f^{\beta}(q)}{\partial q}$.

Let us define two variables, call them A and B :

$$\begin{aligned} A &= \mu_{\alpha} + w_j \\ B &= \mu_{\beta}. \end{aligned} \quad (9)$$

We will discuss the meaning of these two variables further below. Lemma 1 states that the sign of the SOC depends on the relative magnitude of A and B .

Lemma 1 q_j^0 in (6) is the maximum for (4) if and only if $A > B$.

Therefore, when $A > B$ and $q_j^0 \in (q^s, m)$, we have an interior maximum: the agent prefers a supermajority (this case is shown in figure 2). If $A > B$ and $q_j^0 \notin (q^s, m)$, then maximization yields a corner solution: j prefers either unanimity if $q_j^0 \geq m$, or the simple majority if $q_j^0 \leq q^s$.

¹⁴ $q^s = \lceil \frac{m}{2} \rceil$ if m is odd (where $\lceil \frac{m}{2} \rceil$ represents the rounding of $\frac{m}{2}$ up to the integer) and $q^s = \frac{m}{2} + 1$ if m is even.

If $A \leq B$, the optimal threshold is always in the corner: it is unanimity if $EU_j(L_j(q^s)) \leq EU_j(L_j(m))$; otherwise it is the simple majority (see figure 3).¹⁵

Interestingly, the bare majority and unanimity are quite likely to be the most preferred threshold: if $A > B$, as the stationary point is possibly not interior; if $A < B$, as q_j^0 is a minimum.

Let us now look at A and B , which play an important role in j 's decision problem. From (9), B is the mean, μ_β , of the “losing” density, $f^\beta(\cdot)$; while A is the mean, μ_α , of the “winning” density *plus* j 's votes. Recall that $f^\alpha(\cdot)$ and $f^\beta(\cdot)$ have the same variance, thus they are horizontal translations one of another, and their horizontal distance reflects the degree of optimism, p . For example, if j has a pessimistic view about the others' vote ($p < 0.5$), $f^\beta(\cdot)$ is on the right with respect to $f^\alpha(\cdot)$, and $\mu_\beta > \mu_\alpha$. Observe, however, that this does not necessarily imply that j thinks that losing is more likely than winning. In fact, she also takes her own votes into account. The idea is simple: if she has a lot of votes she is “confident” about winning, even when she is pessimistic about how the others will vote. It is easy to see that if $\mu_\alpha + w_j > \mu_\beta$ then $\Pr\{\alpha\} > \Pr\{\beta\}$, for any q . Thus, as soon as $A > B$, the probability of winning is always higher than the probability of losing. Consequently, we can consider the relative magnitudes of A and B as a measure of j 's degree of confidence about the voting outcome.

Definition 1 *Agent j is confident when $A > B$. She is non-confident when $A < B$.*

What does make an agent confident? First, her degree of optimism: $p \geq 0.5$ is a sufficient condition for confidence. Second, her relative voting weight, w_j : if she is sufficiently powerful, she may be confident despite $p < 0.5$.¹⁶ The latter case of pessimism “turned into” confidence by means

¹⁵We assume for simplicity that in the case unanimity and the simple majority give the same expected utility, j chooses unanimity. Moreover, observe that when $A = B$ we do not have a stationary point. We will provide more details on this case in a footnote below.

¹⁶More precisely,

$$\frac{w_j}{2 \sum_{i \in N \setminus j} w_i} > 0.5 - p \Leftrightarrow A > B$$

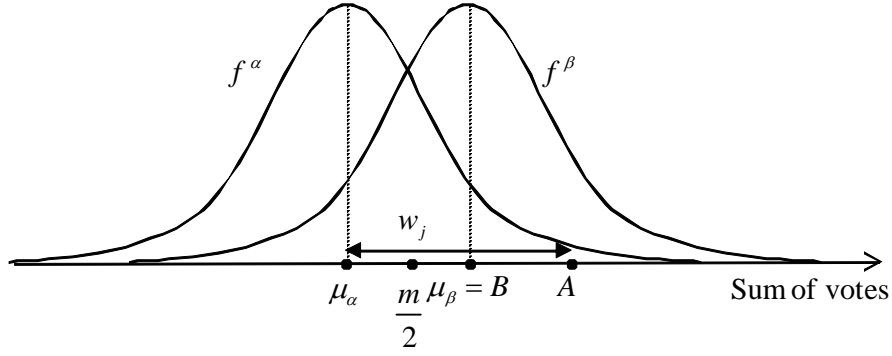


Figure 1: Pessimism, but confidence

of voting power is illustrated in figure 1.

Summing up, confidence is somehow a more comprehensive concept than optimism, because it takes into account, not only priors about how the others vote, but also the pivotal role that an agent might be able to play thanks to her votes.¹⁷ Below, we use the concept of confidence, together with *RASQ*, to analyze how an individual chooses her most preferred threshold.

2.1.1 Confident agent, $A > B$

When agent j is confident, the stationary point is a maximum. The Proposition below shows how the preferred threshold is related to *RASQ*.

Proposition 1 *If $A > B$ and:*

- a) if $RASQ_j > 1$, then agent j prefers a supermajority or unanimity;*
- b) if $RASQ_j \leq 1$, then agent j prefers the simple majority.*

¹⁷With respect to a general idea of optimism, as in Buchanan and Faith (1980, 1981) or in Zorn and Martin (1986), confidence seems more appropriate to analyze contexts in which individuals have a certain degree of influence. For example, legislatures or boards in which differently sized factions are clearly identifiable.

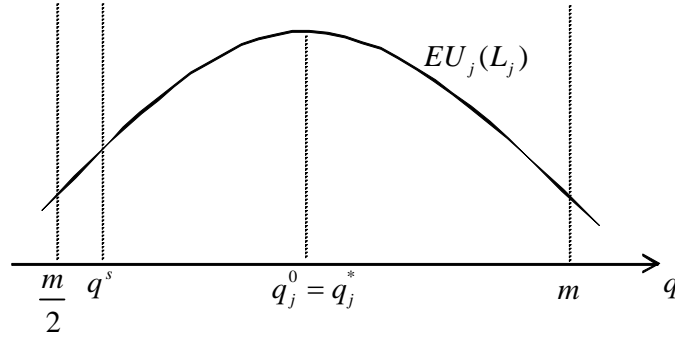


Figure 2: A confident j prefers a supermajority

Let us first discuss part b) of the Proposition. Recall that j thinks that, for any threshold, winning is more likely than losing. Then she wants an easy majority formation because she thinks that a favorable outcome is quite likely. Moreover, if $RASQ_j \leq 1$, the voting prospect presents an additional advantage to her: the cost of losing is (weakly) lower than the advantage of winning. In this case she prefers the lowest possible threshold: the bare majority.

Part a) of the Proposition suggests that there is a trade-off: the higher chance of winning due to confidence is contrasted by the relatively high cost of losing, due to $RASQ_j > 1$. There is demand for protection, thus a supermajority, if not unanimity, is preferred. Figure 2 illustrates the case in which the best threshold is a supermajority.

2.1.2 Non-confident agent, $B > A$

When agent j is non-confident, q_j^0 is a minimum, thus we can only have a corner solution: either unanimity or the simple majority (see figure 3). Unanimity is preferred if the cost of the tyranny is high compared to the advantage of winning. In fact, Proposition 2 states that $RASQ_j \geq 1$ is sufficient for preferring unanimity.

Proposition 2 *If $B > A$ and $RASQ_j \geq 1$ then agent j always prefers unanimity.*

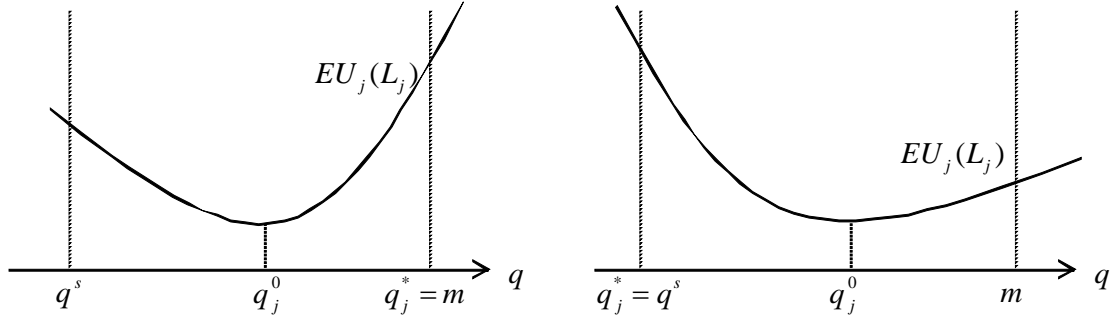


Figure 3: A non-confident j prefers either unanimity or the simple majority

Thus even a small degree of non-confidence, jointly with a small positive difference between the cost of losing and the advantage of winning, induce agent j to prefer the full protection of unanimity (left-hand graph in figure 3).

When does a non-confident agent prefer the simple majority (right-hand graph in figure 3)? Proposition 3 provides a simple answer: since losing is more likely than winning, the relative advantage of winning must be “sufficiently” large. Surprisingly, “sufficiency” corresponds to a quite simple condition: the relative advantage of the status quo must be lower than the ratio between winning and losing probabilities under the simple majority.

Proposition 3 *If $B > A$ and $RASQ_j < 1$, then agent j prefers the simple majority to unanimity if*

$$RASQ_j < \frac{x_\alpha}{1 - x_\alpha} \quad (10)$$

where $x_\alpha = \Pr\{\alpha|_{q=q^s}\}$ is the probability of winning when the simple majority threshold q^s is set up.

According to Proposition 3, one may say that, given two non-confident agents with same priors and power, if the one with the lowest $RASQ$ prefers unanimity, the other one cannot prefer the simple majority. In the next Section, we relate $RASQ$ with risk aversion and we use this argument

to see how agents with different degrees of risk aversion choose their optimal thresholds.¹⁸

2.2 Risk Aversion

We can analyze the effect of risk aversion by looking at $RASQ_j$. A more risk averse agent gets higher utility from avoiding tyranny (the numerator in $RASQ_j$) and lower utility from winning (the denominator). Thus $RASQ_j$ is positively related to risk aversion.

Lemma 2 *Given the monetary values of α , β and ς , $RASQ_j$ is positively related to agent j 's degree of risk aversion.*

Then we can use the relative advantage of the status quo as a measure of an agent's attitude towards risk. Lemma 3 below states that $RASQ_j$ and the stationary point, q_j^0 , move in the same direction when j is confident; they go in opposite directions if j is non-confident. We use these two Lemmas together in Proposition 4 to get a positive relationship between risk aversion and the most preferred majority threshold.

Lemma 3 q_j^0 in (6) is positively related to $RASQ_j$ if and only if $A > B$.

Based on Lemmas 2 and 3, we can show how risk aversion affects preferences for majority rules.

Proposition 4 *All other things being equal, agent j 's most preferred threshold is (weakly) positively related to her degree of risk aversion.*

The result of Proposition 4 is quite intuitive: a more risk averse agent prefers a higher threshold because majority formation can be blocked more easily. She gets more protection, in exchange for

¹⁸Consider the case in which $A = B$. We already know that there is no stationary point. However, it is easy to see that, if $RASQ_j > 1$, then EU_j is always decreasing. Thus j prefers the simple majority. If $RASQ_j < 1$, then she prefers unanimity.

the chance of getting a favorable outcome. In other words, a risk averter has a conservative attitude towards political changes. This attitude is revealed by a stronger preference for less decisive voting rules.

2.3 Voting weight

Consider now voting weight, w_j . More voting power gives more control over the decision, so that the outcome is more likely to be the preferred policy. All other things being equal, agents with higher voting power have stronger preferences for lower thresholds.

Proposition 5 *The most preferred threshold is (weakly) negatively related to w_j .*

Thus the powerful want to facilitate majority formation by lowering the threshold. They are less conservative because they have higher chance to tyrannize others and lower risk of being tyrannized.¹⁹

3 The Constitutional game

We now introduce a constitutional stage. Agents agree that voting is the way in which future conflicts between majority and minority will be solved. A justification is that making an agreement today on how to make future decisions is more efficient than bargaining on every single decision in the future. This is consistent with reality and with a common approach to constitutions as incomplete contracts (Persson and Tabellini, 2000; Aghion and Bolton, 2002). We describe constitutional

¹⁹Observe that the voting power has the same impact as optimism on outcome probabilities. One could even say that power allows an agent to “afford” a certain degree of pessimism about the other agents’ behavior.

In fact, it is easy to see that, if w_j is high enough to give player j veto power under q^s , her probability of losing is zero for any q . Thus, she can either win or stay in the status quo. In this extreme case, independently of both her gains and losses and her degree of optimism, she always prefers the simple majority.

negotiations as a bargaining over voting rules, namely the majority threshold. The reason why we model constitutional negotiations as a unanimous bargaining game is that at the very origin of any constitution there must be a “voluntary exchange” in which anyone accepts to establish a way to make common decisions. Thus the primitive must be a unanimous agreement.²⁰

A standard assumption in the constitutional analysis is that individuals are behind a veil of ignorance: they are unaware of any difference amongst them. If this is the case, the issue of constitutional negotiations is empty: everyone agrees on the same voting rule. A non-trivial analysis of constitutional negotiations implies a certain degree of heterogeneity amongst agents.²¹ In our perspective, heterogeneity may concern risk aversion, degree of optimism, and voting power, or any subset of them. While risk aversion is a subjective attitude, degree of optimism or voting power may reflect objective and stable differences in the constituents’ original positions. Consider, for example, negotiations on a federal constitution and assume that many regions in the country are rich and industrialized. Reasonably, a poor unindustrialized region feels pessimistic *ex ante* that in the future any other region will vote in a favorable way. Thus, that region pushes for a less decisive constitution. Assume also that regions are different in size. A small region is possibly aware that

²⁰This is consistent with the approaches to constitutions of Wicksell, Lindhal, Musgrave, and many others. See Mueller (2003), for an excellent discussion.

An alternative modelling choice is formal voting on q at the constitutional stage (as in Messner and Polborn, 2004). The outcome is simple: since preferences on q are single peaked, there is at least one Condorcet winner, which is the median’s q if constitutional voting adopts the simple majority. With formal constitutional voting, however, the socially preferred q does not reflect the preferences of the minority. Thus it is not clear why the minority should accept to sign the constitution. The constitutional table is typically the place where minorities try to defend their rights. Realistically, our modelling choice with unanimous bargaining seems more appropriate than formal voting on q .

²¹ “Constitutions are not written by social planners, and veils of ignorance have holes in them.” (Aghion, Alesina and Trebbi, 2004, p. 578).

in any future decisions it will have low voting power. Arguably, during constitutional negotiations this region will insist on protective rules.

We model the constitutional stage as a Nash bargaining game over a “material” outcome, which is in this case the majority threshold, q . In addition to what we pointed out in the Introduction, there are at least two appealing features of this modelling choice. First, Nash bargaining requires unanimous consensus. This aspect properly captures both the cooperative and the non cooperative elements of constitutional negotiations. Unanimity, in fact, exposes negotiators to an implicit trade-off. On the one hand, it enhances the decisiveness of each negotiator: since no decision can be taken at the expenses of weak minorities, any valid proposal must adequately represent the interests of all negotiators. On the other hand, given the high costs of a failure, there is no incentive to adopt purely obstructionist strategies. Second, due to the neutrality and reasonability of its axioms, Nash bargaining can be adopted as a fair arbitration scheme, that satisfies basic criteria of impartiality in distributive justice (Mariotti, 1999).

The choice of q at the constitutional stage will design the voting lottery of the legislative one. The payoff vector in the constitutional bargaining is the profile of the agents’ expected utilities attached to the lotteries generated by q : $\{EU_1(L_1(q)), \dots, EU_n(L_n(q))\}$, where $EU_j(L_j(q))$ is defined by (4); $j = 1, \dots, n$.²²

Call $\Sigma \subset \mathbb{R}^n$ the feasible set of the bargaining problem. Let ϑ be the fallback option in the constitutional stage. The disagreement point, $u(\vartheta) = \{u_1(\vartheta), \dots, u_n(\vartheta)\}$, is the vector of the

²²Typically, constitutions design voting rules for “many” future legislative decisions. Thus, our previous analysis of the legislative lottery applies if we consider that α and β are not alternative proposals regarding a specific issue, but rather alternative future platforms or reforms in several different fields of the public life. Since at the constitutional stage there might be little knowledge about that, we can simply assume that gains and losses are opposed and equally sized values, say $\alpha = 1$, $\beta = -1$, and $\varsigma = 0$. This implies that at the constitutional level the only sources of heterogeneity are risk aversion, optimism and perceived power.

individual utilities in ϑ .²³ Thus, our constitutional bargaining problem is the pair $(\Sigma, u(\vartheta))$. The agents bargain over the threshold space $[q^s, m]$; each threshold maps into Σ , yielding agents' payoffs. For sake of simplicity, we will refer to the Nash Bargaining Solution (NBS henceforth) as the equilibrium threshold, q^N , rather than the payoff vector that it generates. The NBS threshold q^N is a point in $[q^s, m]$ that generates the payoff vector which maximizes the Nash product:

$$\max_{q \in [q^s, m]} \prod_{j=1}^n [EU_j(L_j(q)) - u_j(\vartheta)] \quad (11)$$

3.1 Existence and unicity

By definition, q^N must ensure that all agents are assigned (in expected value) at least as much as they can get from the fallback option, i.e. $EU_j(L_j(q)) - u_j(\vartheta) \geq 0$, for all j . Call $\mathbf{Q} \subseteq [q^s, m]$ the set of thresholds such that all agents' payoffs are higher than the disagreement point. Of course q^N must belong to \mathbf{Q} . A first issue is emptiness of \mathbf{Q} . For simplicity, we assume that $\mathbf{Q} = [q^s, m]$, thus the feasible set Σ is non-empty.²⁴ In the case \mathbf{Q} is empty, the agreement on the constitution will never be reached since there is no threshold that gives each agent a voting lottery that is preferred to the disagreement point. It is easy to see that \mathbf{Q} is likely to be empty as long as, at least for some agents, fallback utilities are high, expected utilities are low, agents' expected utility functions are far apart from one another. If this is not the case, agents reach an agreement on the constitution.

A second issue is unicity of q^N . If the feasible set Σ is convex, q^N exists and it is unique (Nash,

²³Observe that we do not assume that ϑ is necessarily ς . In fact, they might be different: ϑ represents what would occur if the constitutional negotiations break down, whereas ς is what happens if no majority is formed at the legislative stage. Thus, ϑ and ς depend on contingencies that occur at the constitutional stage and the legislative stage respectively. Since such contingencies might be substantially different, we simply do not assume that ς and ϑ are the same.

²⁴Assuming $\mathbf{Q} = [q^s, m]$ means that all agents prefer to reach any agreement on a majority threshold instead of ending up into the disagreement point. This might not be always true. However, this assumption simplifies our analysis. Nonetheless, removing it would not substantially change our findings.

1950). It is easy to see that if all agents' payoff functions are concave in $[q^s, m]$, then Σ is convex. If, at least for one agent j , EU_j is convex in some subset of \mathbf{Q} , then Σ might no longer be convex. In this case, we might have multiple solutions for (11). However, we should be able to rank them, choosing the global maximum amongst several local ones.²⁵

The curvature of the payoff functions plays a relevant role in our problem; below, it will become clear that this role is not only confined to the multiplicity of equilibria, but it also regards j 's ability to reach a favorable agreement (see Lemma 6 and Corollary 1 below). Therefore, it is the case to dwell a bit on the changes in EU_j curvature. From Lemma 1 above, we know that confidence (non-confidence) is sufficient for concavity (convexity) in the stationary point. As for the rest of the threshold space, we may rely on Lemma 4 below, which states that, close to the boundaries of the threshold space, EU_j curvature may change if the agent's "level of confidence/non-confidence" is low. An example of this kind of change in EU_j curvature is shown in figure 5.

Lemma 4 *i) If $A > B$, then EU_j is concave for any $q \in [B, A]$; moreover, EU_j is concave for all $q \in [q_s, m]$ if $A - B$ is large enough.*
ii) If $A < B$, then EU_j is convex for any $q \in [A, B]$; moreover, EU_j is convex in all $q \in [q_s, m]$ if $B - A$ is large enough.

Therefore, multiple solutions are likely when there are some non-confident agents, or when some of the confident ones have either moderate degrees of optimism/pessimism or low voting weights. In fact, in this case, their level of confidence is quite low. Broadly speaking, there is a unique

²⁵The Nash bargaining theory was originally formulated for convex feasible sets. Mariotti (1998a), Mariotti (1998b) and Zhou (2000), by relaxing some of the Nash's axioms, prove that the NBS exists also for non-convex problems; the main features of the original solution hold, except unicity. Denicolò and Mariotti (2000) show that, when the family of admissible non-convex problems is large enough, it is possible to associate a social welfare ordering to that family.

solution when *all* constituents are rather confident; i.e. either they share similar preferences, or those with special interests have also consistent voting power. Otherwise, we have multiple local maxima. Below we show that in the latter case, small changes in the parameters may cause huge shifts in the equilibrium threshold.

3.2 The constitutional bargaining: voting power, optimism and risk aversion

The constitutional agreement reflects individual preferences about q , and ultimately all factors that affect them. Below, we focus on three factors: individual level of optimism, degree of risk aversion and the voting weight.

Consider an agent j . Her power in the negotiation arena derives from her ability to “pull” q^N towards a threshold that she likes more, contrasting the tendency of others to pull it in the opposite direction. In a Nash bargaining context, j has this kind of bargaining power when her payoff is low compared to the the rest of the group, and when the increase in the payoff due to shifting q towards her most preferred threshold is high compared to the loss suffered by the rest of the group. In fact, it is easy to see that the Nash product increases if the ratio between j ’s payoff and its derivative is lower than the ratio between the product of the rest of the agents’ payoffs and the derivative of the latter product.²⁶ Broadly speaking, the Nash product can be studied by looking at the absolute value of these two ratios; we may even say that j ’s power in negotiations derives from a discrepancy between them. Our intuition is that, if any changes in j ’s optimism, risk aversion or voting weight cause the former ratio to be lower than the latter one, then j becomes more powerful and the equilibrium threshold moves towards j ’s most preferred threshold.

We can prove this intuition in two steps. First, we introduce Lemmas 5 and 6 that concern the

²⁶The reader may observe that the FOC of the Nash product maximization in (16) in Appendix 4.1 is related to this idea.

marginal impact of vertical and horizontal shifts in EU_j on the equilibrium agreement.²⁷ Then, we use these Lemmas to support Proposition 6, which will look at how the NBS is affected by changes in j 's level of optimism, risk aversion and voting weight.

Lemma 5 *A downward shift of EU_j decreases the equilibrium threshold if $EU'_j(q^N) < 0$, and increases it otherwise.*

Let us discuss this Lemma. When EU_j shifts downwards, the ratio between j 's payoff and its derivative gets lower at any threshold. According to our idea of bargaining power, j becomes more powerful. In fact, by Lemma 5, the equilibrium agreement moves towards j 's most preferred threshold. Broadly speaking, when j receives less utility from any possible agreement, then she gets “compensated” by a more favorable NBS. For example, if $q_j^0 < q^N$ and agent j is non-confident (see figure 4), the equilibrium threshold moves upwards, in a direction that yields a higher payoff for j . This case with non-confidence has an interesting feature; let us expatiate a bit on it. Observe that j 's most preferred threshold is the bare majority. However, q^N is “close” to unanimity, which can be considered a sort of second best for j . Presumably, we have multiple NBS, and q^N is the global maximum. Interestingly, if EU_j shifts downwards, despite j is more powerful the agreement goes up towards j 's second best. Our intuition is that j uses her power to get a “feasible” improvement in the agreement: the simple majority is too far away and pushing towards it would produce a worse result.

The following Lemma considers a horizontal shift in EU_j .

Lemma 6 *If EU_j shifts rightward,*

i) q^N always increases if $EU''_j(q^N) < 0$;

²⁷Our analysis of vertical and horizontal shifts of EU_j concerns *marginal* shifts. This means that, if the feasible set is non-convex and we have multiple local maxima, the “new” global maximum for the Nash product remains in the neighbourhood of q^N .

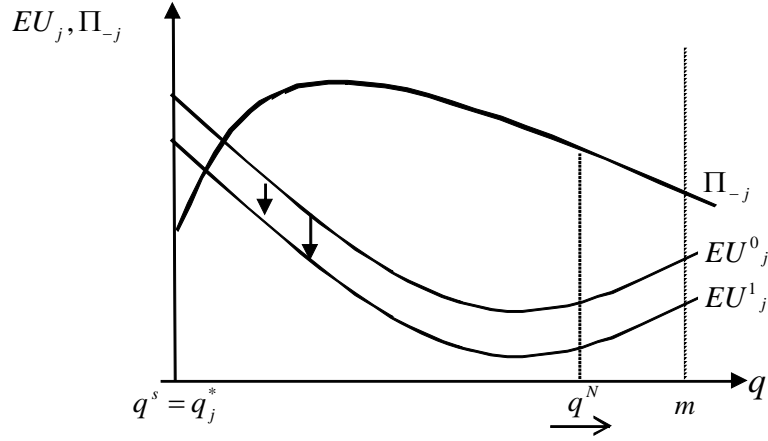


Figure 4: q^N raises towards a second best

ii) q^N increases if $EU_j''(q^N) > 0$ and $\left| \frac{EU_j''(q^N)}{EU_j'(q^N)} \right|$ is “low enough”.

Let us interpret this Lemma. Consider the most comprehensive case in part ii) of the Lemma. Assume that EU_j is increasing and convex in q^N . Also assume that $q_j^* = q_j^0$ and $q^N < q_j^0$ (see figure 5). From the FOC of the Nash product in equation (16) of Appendix 4.1, we have:

$$EU_j' \cdot \Pi_{-j} = EU_j \cdot (-\Pi_{-j}') \quad (12)$$

This equation says that in equilibrium the *marginal increase* in the Nash product due to a marginal increase in j 's payoff (the left-hand side) must equal the *marginal reduction* in the product due to a marginal decrease in the other players' payoffs (the right-hand side). Consider now a rightward shift of EU_j . We have two effects. First, the left-hand side lowers because EU_j' decreases (in the figure, EU_j is flatter in q^N). Second, the right-hand side lowers as well (EU_j shifts downwards). If the first effect is stronger, then q^N is too low: an increase in the threshold causes the Nash product to raise.

Summing up, if EU_j shifts rightward and it is concave, the equilibrium threshold raises in

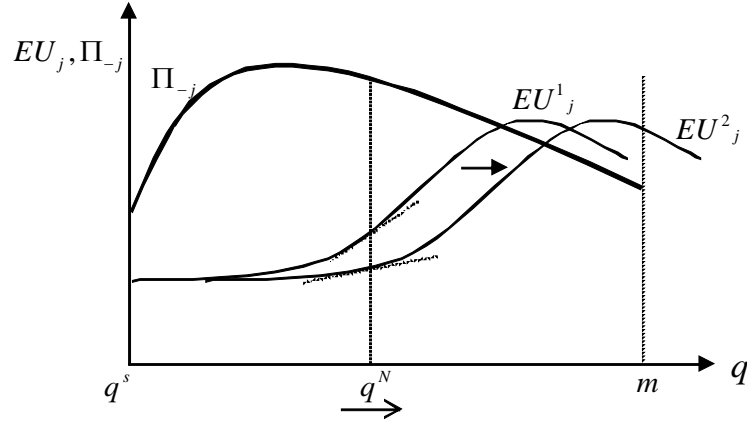


Figure 5: Convex and increasing EU_j , with a threshold rise

any case. If EU_j is convex (like in figure 5), the agreement is a higher threshold only if $\left| \frac{EU_j''}{EU_j'} \right|$ is sufficiently low (inequality (20) in Appendix 4.1 holds). The reader may observe that, for $EU_j' > 0$, this ratio is the inverse of the Arrow-Pratt measure of absolute risk aversion (Arrow, 1965; Pratt, 1964). This suggests an interesting relation between our approach to Nash bargaining and the general idea embodied in the Arrow-Pratt coefficient.²⁸

Corollary 1 *If the Arrow-Pratt coefficient in $EU_j(q^N)$ is sufficiently high, a rightward shift in EU_j allows agent j to get a more favorable agreement.*

The general idea is that agent j 's power in constitutional negotiations is related to the curvature and the position of her payoff function, EU_j . From Section 2, we know that the primitives that determine curvature and position are risk aversion, optimism and voting weight. Proposition 6 states that j is more powerful in the constitutional arena when she perceives to be less powerful at the legislative stage, when she is more risk averse, and when she feels less optimistic about how

²⁸A warning is needed. Observe that here we talk about the Arrow-Pratt coefficient only with reference to the curvature of j 's payoff function. Thus, we do not refer to this coefficient as a measure of j 's risk attitudes. In fact, there is no uncertainty specifically related to constitutional bargaining.

the others will vote in the future.

Proposition 6 *i) If EU_j is concave in q^N , the NBS threshold is more favorable to j if one or more of the following changes occur: a) w_j decreases; b) p decreases; c) j 's risk aversion increases.*
ii) If EU_j is convex in q^N , the statement above holds if the Arrow-Pratt coefficient in $EU_j(q^N)$ is sufficiently high.

Proposition 6 synthesizes the rationale behind our application of Nash bargaining to constitutional negotiations: assembly members who expect lower utility from future voting are rather powerful when the decision concerns the voting rules. Our model suggests that this kind of bargaining power depends on three factors: aversion to the risk of being tyrannized in future voting; low degree of optimism about how the others' will vote in the future; low degree of influence on future decisions. The last factor sounds somehow paradoxical: those who anticipate low leverage in the future legislative phase are rather powerful in the constitutional phase. That is to say that in bargaining the winners are those who have the most at stake.

4 Conclusion

In this paper we described voting from the perspective of an individual who is uncertain about how the others will vote. Probabilities of favorable or unfavorable outcomes depend on voting rules. We focused on the majority threshold, which is a measure of the decisiveness/protectiveness of a voting system. We showed that an individual prefers a higher threshold when she is more risk averse, less powerful or less optimistic about the likelihood that others will vote like her. De facto, raising the threshold is a form of protection against the risk of being tyrannized by an unfavorable majority. We found that confidence about votes of others is a crucial behavioral parameter, and that the preferred majority threshold sometimes only takes on extreme values, the bare majority

or unanimity.

We used these preferences over majority thresholds to build a constitutional game. A constitutional assembly forms because reaching an early agreement on how to make future decisions is more efficient than bargaining on every single decision in the future. Within the assembly, conflict on decisional rules is hardly reconcilable with the veil of ignorance hypothesis. Thus we assumed that assembly members are heterogeneous in the perception of the tyranny that they will possibly suffer in the future. This assumption is consistent with several real life examples. So the richest colonies at the time of the Philadelphia Convention were aware of a higher risk of being expropriated; large EU member states were not agnostic about their future power when they were negotiating for the EU Constitution. We adopted the Nash bargaining solution for the constitutional game since this solution represents a general and fair arbitration scheme. At the negotiation table, risk averse members and small minorities perceive a high risk of tyranny. In a sense, they have the most at stake. This makes them tougher during constitutional bargaining. Thus they succeed in getting a high level of protection. Our model illustrates that fair constitutional negotiations amongst “unequals” represent the chance for the weak and the diverse ones to get protection against the tyranny of majority that democratic processes entail.

As stated in the Introduction, there is no bargaining at the legislative stage. No compensatory payments or proposal adjustments are possible in order to collect more votes. Proposals and types are exogenous, with a significant simplification of the analysis. Allowing for legislative bargaining in which proposals are endogenous would be an interesting extension, but it would possibly require a more complex setting. Bargaining procedures should be fully specified.²⁹ The idea would be that, through side-payments, a formateur collects the “yes” votes of other agents and forms a majority.

²⁹Full specification is also needed to overcome the problem of multiple Condorcet winners when a supermajority is adopted (Black, 1948).

Suppose that the supermajority raises. The formateur has to pay more people in order to collect a wider support (as, for example, in Ansolabehere, Snyder and Ting, 2003). An agent who is initially in the minority has a chance to get payments if she casts her vote. This chance makes the expected outcome less “tyrannical”. In a sense, because of side-payments the expected tyranny is less severe. Our hunch is that in this case risk averse or weak agents might still prefer a supermajority, but possibly lower than the one predicted in this paper. A likely result is that whenever side-payments or other efficient forms of legislative bargaining are possible, people would agree on a more decisive constitution.

Finally a caveat. An important assumption in this paper is that there is a sufficiently large number of agents in the society (more than twenty). This allowed us to build continuous preference functions with the benefits of differential calculus, weighted votes analysis, and a consistent number of general results. In Appendix 4.2 we drafted a discrete model that applies to small groups. This model yields fewer results, which are broadly the same as in the continuous model. Still some of them hold if the number of agents is sufficiently large. If we remove this assumption our findings, both with the continuous and with the discrete model, should be interpreted as broad approximations of what is likely to happen.

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Appendix

4.1 Proofs of Propositions and Lemmas

Proof. Lemma 1. Rearranging (5) yields

$$\frac{f^\alpha(q_j^0 - w_j)}{f^\beta(q_j^0)} = RASQ_j \quad (13)$$

Substituting (13) for $RASQ_j$ into (8) yields, under the assumption that $EU_j(L_j(q))$ is concave in q ,

$$f_q^{\beta'}(q_j^0) \cdot \frac{f^\alpha(q_j^0 - w_j)}{f^\beta(q_j^0)} < f_q^{\alpha'}(q_j^0 - w_j),$$

from which

$$\frac{f_q^{\beta'}(q_j^0)}{f^\beta(q_j^0)} < \frac{f_q^{\alpha'}(q_j^0 - w_j)}{f^\alpha(q_j^0 - w_j)}.$$

Recall that the $f(\cdot)$'s are normal distributions; thus:

$$-\frac{1}{\sigma^2}(q_j^0 - \mu_\beta) < -\frac{1}{\sigma^2}(q_j^0 - w_j - \mu_\alpha)$$

or,

$$w_j + \mu_\alpha > \mu_\beta$$

from which the Lemma follows. ■

Proof. Proposition 1. From $A > B$ and Lemma 1, we know that q_j^0 is a maximum.

a) From (6), we can see that $q_j^0 > q^s$ only if $\frac{\sigma^2 \ln RASQ_j}{w_j + \mu_\alpha - \mu_\beta} > 0$. Observe that the denominator in this inequality is $A - B$, and it is positive. This implies that $RASQ_j$ must be higher than one. Therefore, $q_j^* \in (q^s, m]$.

b) Following the same argument, we conclude that $q_j^0 \leq q^s$ if $RASQ_j \leq 1$. ■

Proof. Proposition 2. From $B > A$ and Lemma 1, we know that q_j^0 is a minimum. From (6) we can see that $q_j^0 \leq q^s$ only if $\frac{\sigma^2 \ln RASQ_j}{w_j + \mu_\alpha - \mu_\beta} \leq 0$. By $B > A$ the denominator in this inequality is

negative, and by $RASQ_j \geq 1$ the numerator is non-negative. Thus EU_j is monotonically increasing in $[q^s, m]$. Consequently, $q_j^* = m$. ■

Proof. Proposition 3. From $B > A$ and Lemma 1, we know that q_j^0 is a minimum. Moreover, from (6) and $RASQ_j < 1$, we know that $q_j^0 > q^s$. Agent j has to decide whether she is better off under the simple majority or under unanimity. Outcomes α and β occur with probability $x_\alpha = \Pr \{\alpha | q=q^s\}$ and $1 - x_\alpha$, respectively. From $B > A$, it follows that $x_\alpha < 0.5$. Recall that under unanimity the status quo is (almost) certain; with the simple majority, the status quo is impossible. Thus, the expected utilities of voting under unanimity and under the simple majority are $u_j(\varsigma)$ and $u_j(\alpha) \cdot x_\alpha + u_j(\beta) \cdot (1 - x_\alpha)$, respectively. Agent j prefers simple majority to unanimity if

$$u_j(\alpha) \cdot x_\alpha + u_j(\beta) \cdot (1 - x_\alpha) > u_j(\varsigma).$$

By rearranging,

$$[u_j(\alpha) - u_j(\varsigma)] \cdot x_\alpha > [u_j(\varsigma) - u_j(\beta)] \cdot (1 - x_\alpha)$$

then,

$$x_\alpha > RASQ_j \cdot (1 - x_\alpha)$$

from which the proposition follows. ■

Proof. Lemma 2. We prove this Lemma by comparing two agents who get the same monetary payoff from the same policies, but differ in their risk attitudes. Let $g_i : \{\alpha, \beta, \varsigma\} \rightarrow \mathbb{R}$ be a function that assigns the monetary values of policy outcomes to any agent i , ($i = 1, \dots, n$). Consider two α -type agents, r and s in N . Assume that $g_r = g_s$. Let us write, for simplicity, $u_r(g_r(\cdot)) = u_r(\cdot)$ and $u_s(g_s(\cdot)) = u_s(\cdot)$. Suppose that r is more risk averse than s . Thus, an increasing and concave function $t : \mathbb{R} \rightarrow \mathbb{R}$ exists such that $u_r(x) = t(u_s(x))$ for each $x \in X$, where X is a closed interval in \mathbb{R} . Therefore, we can write:

$$RASQ_r = \frac{u_r(\varsigma) - u_r(\beta)}{u_r(\alpha) - u_r(\varsigma)} = \frac{t(u_s(\varsigma)) - t(u_s(\beta))}{t(u_s(\alpha)) - t(u_s(\varsigma))}.$$

We want to prove that

$$RASQ_r > RASQ_s.$$

Since $u_s(\varsigma) \in (u_s(\beta), u_s(\alpha))$, it can be rewritten as a convex linear combination of $u_s(\beta)$ and $u_s(\alpha)$, i.e. $u_s(\varsigma) = a \cdot u_s(\alpha) + (1 - a) \cdot u_s(\beta)$, with $a \in (0, 1)$. Thus we can write

$$RASQ_s = \frac{[a \cdot u_s(\alpha) + (1 - a) \cdot u_s(\beta)] - u_s(\beta)}{u_s(\alpha) - [a \cdot u_s(\alpha) + (1 - a) \cdot u_s(\beta)]} = \frac{a}{1 - a}$$

and

$$RASQ_r = \frac{t([a \cdot u_s(\alpha) + (1 - a) \cdot u_s(\beta)]) - t(u_s(\beta))}{t(u_s(\alpha)) - t([a \cdot u_s(\alpha) + (1 - a) \cdot u_s(\beta)])}.$$

By the concavity of $t(\cdot)$, we know that $t[a \cdot u_s(\alpha) + (1 - a) \cdot u_s(\beta)] > a \cdot t(u_s(\alpha)) + (1 - a) \cdot t(u_s(\beta))$.

Thus we can write

$$RASQ_r > \frac{a \cdot t(u_s(\alpha)) + (1 - a) \cdot t(u_s(\beta)) - t(u_s(\beta))}{t(u_s(\alpha)) - [a \cdot t(u_s(\alpha)) + (1 - a) \cdot t(u_s(\beta))]} = \frac{a}{1 - a}$$

or

$$RASQ_r > RASQ_s.$$

■

Proof. Lemma 3. We want to prove that

$$\frac{\partial [RASQ_j]}{\partial q_j^0} > 0 \tag{14}$$

Recall that rearranging (5) yields:

$$\frac{f^\alpha(q_j^0 - w_j)}{f^\beta(q_j^0)} = RASQ_j$$

where

$$\frac{f^\alpha(q_j^0 - w_j)}{f^\beta(q_j^0)} = e^{\frac{(q_j^0 - \mu_\beta)^2 - (q_j^0 - w_j - \mu_\alpha)^2}{2\sigma^2}}.$$

Then satisfying (14) implies that

$$\frac{\partial \left[\frac{f^\alpha(q-w_j)}{f^\beta(q)} \right]}{\partial q} = \frac{1}{\sigma^2} (w_j + \mu_\alpha - \mu_\beta) \cdot e^{\frac{(q_j^0 - \mu_\beta)^2 - (q_j^0 - w_j - \mu_\alpha)^2}{2\sigma^2}} > 0$$

which, given that $\sigma^2 > 0$ and $e(\cdot) > 0$, is satisfied if and only if

$$w_j + \mu_\alpha > \mu_\beta$$

where $w_j + \mu_\alpha = A$ and $\mu_\beta = B$. ■

Proof. Proposition 4. Let us distinguish two sub-cases: a) agent j is confident ($A > B$); b) agent j is non-confident ($B > A$).

Let us first consider sub-case a). From Lemmas 2 and 3 we know that when agent j is confident q_j^0 increases in her risk aversion. Moreover, from Proposition 1 we know that if $RASQ_j > 1$, then she prefers either a supermajority or unanimity. Therefore, as a consequence of an increase in risk aversion we can have three situations: 1. $RASQ_j$ increases, but it is still not larger than one; 2. $RASQ_j$ increases and becomes larger than one; 3. $RASQ_j$ is already larger than one, and it increases. In Case 1, agent j continues to prefer the simple majority. In Case 2, she stops preferring the simple majority and switches in favor of a supermajority (or unanimity). In Case 3, she prefers a higher majority threshold (or unanimity). Thus, the preferred majority threshold never decreases. The proof of sub-case b) works in the opposite way. From Lemmas 2 and 3, we know that if agent j is non-confident then a higher risk aversion affects q_j^0 negatively. From Lemma 1 we know that she can only prefer either the simple majority or unanimity; Proposition 2 and 3 state that she prefers the simple majority only if $RASQ_j < 1$ and inequality (10) is satisfied. Therefore, if $RASQ_j \geq 1$ and risk aversion increases, she continues to prefer unanimity. If $RASQ_j < 1$ and risk aversion increases, we can have three cases: 1. If $RASQ_j$ increases, but inequality (10) is still satisfied, she keeps preferring the simple majority; 2. If $RASQ_j$ increases and inequality (10) is no longer satisfied, she shifts from the simple majority to unanimity; 3. If inequality (10) is not satisfied

and $RASQ_j$ increases, she continues to prefer unanimity. Then, also in sub-case b), the preferred threshold cannot decrease in risk aversion. ■

Proof. Proposition 5. Suppose that w_j increases from w_j^1 to w_j^2 , with $w_j^1 < w_j^2$. We can have three cases.

Case 1. $A(w_j^1) > B$ and $A(w_j^2) > B$ (j is confident both before and after the increase in her voting weight). If $RASQ_j \leq 1$, then by Proposition 1, j continues to prefer the simple majority. If $RASQ_j > 1$, then by Proposition 1, agent j prefers a lower majority threshold.

Case 2. $A(w_j^1) < B$ and $A(w_j^2) > B$ (j is non-confident before the increase in her voting weight and becomes confident afterwards). If $RASQ_j < 1$, then by Propositions 1 and 2, either j continues to prefer the simple majority if, before the increase in w_j , inequality (10) was satisfied, or she switches from unanimity to the simple majority if inequality (10) was not satisfied before the increase in w_j . If $RASQ_j = 1$, then by Propositions 1 and 2 j switches from unanimity to the simple majority. If $RASQ_j > 1$, then by Proposition 2 the agent j switches from unanimity to a supermajority.

Case 3. $A(w_j^1) < B$ and $A(w_j^2) < B$ (j is non-confident both before and after the increase in weight). If $RASQ_j \geq 1$, then by Proposition 2, agent j continues to prefer unanimity. If $RASQ_j < 1$ then by Proposition 2 and 3, if inequality (10) was not satisfied before the increase in weight, either agent j continues to prefer unanimity if, after the increase in w_j , inequality (10) is still not satisfied, or she switches from unanimity to the simple majority if inequality (10) is satisfied after the increase in w_j . If inequality (10) was satisfied before the increase in w_j , j continues to prefer the simple majority after the increase in w_j .

Then, in all cases the most preferred threshold cannot increase in the voting weight. ■

Proof. Lemma 4. Recall that $f^\alpha(\cdot)$ and $f^\beta(\cdot)$ are normal density functions with specific

parameters. Thus, we can write:

$$\frac{\partial^2 EU_j(L_j(q))}{\partial q^2} = \frac{(q-A) \cdot e^{-\frac{(q-A)^2}{2\sigma^2}} - RASQ_j \cdot (q-B) \cdot e^{-\frac{(q-B)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma^3 [u_j(\alpha) - u_j(\varsigma)]}$$

where $A = \mu_\alpha + w_j$ and $B = \mu_\beta$. This second-order derivative is positive if and only if:

$$(q-A) \cdot e^{-\frac{(q-A)^2}{2\sigma^2}} > RASQ_j \cdot (q-B) \cdot e^{-\frac{(q-B)^2}{2\sigma^2}} \quad (15)$$

If $A > B$, inequality (15) is never true for $q \in [B, A]$: EU_j is always concave in $[B, A]$. Moreover, if $A < B$, (15) is always true for any $q \in [A, B]$: EU_j is always convex in $[A, B]$.

If $q < \min(A, B)$, (15) becomes

$$\frac{(A-q) \cdot e^{-\frac{(q-A)^2}{2\sigma^2}}}{(B-q) \cdot e^{-\frac{(q-B)^2}{2\sigma^2}}} < RASQ_j$$

and if $q > \max(A, B)$, (15) becomes

$$\frac{(q-A) \cdot e^{-\frac{(q-A)^2}{2\sigma^2}}}{(q-B) \cdot e^{-\frac{(q-B)^2}{2\sigma^2}}} > RASQ_j.$$

If $A > B$, both inequalities above are verified for a sufficiently small $A - B$. Thus, EU_j is convex for any $q \notin [B, A]$ if the level of confidence, as measured by $A - B$, is small.

If $A < B$, both inequalities are verified for a sufficiently large $B - A$. Thus, EU_j is convex for any $q \notin [A, B]$ if the level of non-confidence is large. ■

Proof. Lemma 5. The magnitude of a downward shift of the payoff function, EU_j , can be measured by an increase in a positive parameter b that enters EU_j in the following way: $EU_j = h_j(q) - b$, with $h_j : [q^s, m] \rightarrow \mathbb{R}^+$. Let us simplify the notation by letting $\Pi_{-j} = \prod_{i \in N \setminus j}^n EU_i$, and $\Pi'_{-j} = \frac{\partial \Pi_{-j}}{\partial q}$. Moreover, unless it is differently specified, hereafter EU_j , Π_{-j} and their derivatives are evaluated at q^N .

Recall that q^N satisfies the FOC and the SOC in problem (11). In our simplified notation, the FOC of (11) is:

$$EU'_j \cdot \Pi_{-j} + EU_j \cdot \Pi'_{-j} = 0 \quad (16)$$

and the SOC is:

$$EU''_j \cdot \Pi_{-j} + 2EU'_j \cdot \Pi'_{-j} + EU_j \cdot \Pi''_{-j} < 0 \quad (17)$$

Let us study the sign of $\frac{\partial q^N}{\partial b}$. By implicitly differentiating (16), we get:

$$\frac{\partial q^N}{\partial b} = -\frac{-\Pi'_{-j}}{EU''_j \cdot \Pi_{-j} + 2EU'_j \cdot \Pi'_{-j} + EU_j \cdot \Pi''_{-j}}. \quad (18)$$

The denominator of (18) is the SOC in (17), and it is negative by assumption. As for the numerator, from (16) it is easy to see that Π'_{-j} has the opposite sign of EU'_j . If $EU'_j < 0$, thus $-\Pi'_{-j} < 0$ and $\frac{\partial q^N}{\partial b} < 0$. If $EU'_j > 0$, thus $-\Pi'_{-j} > 0$ and $\frac{\partial q^N}{\partial b} > 0$. ■

Proof. Lemma 6. The magnitude of a rightward shift of j 's expected utility function can be measured by a positive parameter c that enters j 's expected utility in the following way: $EU_j = l_j(q - c)$, with $l_j : \mathbb{R} \rightarrow \mathbb{R}^+$. Let us use the same simple notation as Lemma 5 above. Observe that $\frac{\partial EU_j}{\partial c} = -\frac{\partial EU_j}{\partial q} = -EU'_j$.

We have to prove that $\frac{\partial q^N}{\partial c} > 0$. Let us follow the same rationale as in the proof of Lemma 5; the implicit differentiation of (16) yields:

$$\frac{\partial q^N}{\partial c} = -\frac{-EU''_j \cdot \Pi_{-j} - EU'_j \cdot \Pi'_{-j}}{EU''_j \cdot \Pi_{-j} + 2EU'_j \cdot \Pi'_{-j} + EU_j \cdot \Pi''_{-j}}. \quad (19)$$

The denominator in (19) is the SOC in (17), which is negative. Observe that EU'_j and Π'_{-j} have opposite sign, otherwise (16) would not be satisfied; therefore, $EU'_j \cdot \Pi'_{-j} < 0$.

If EU_j is concave, then the numerator in (19) is always positive. Therefore, $\frac{\partial q^N}{\partial c} > 0$.

If EU_j is convex, $\frac{\partial q^N}{\partial c} > 0$ if $EU''_j \cdot \Pi_{-j} + EU'_j \cdot \Pi'_{-j} < 0$. Observe that the first term in the left-hand side of this inequality is positive and the second one is negative. This inequality can be rewritten as $EU''_j \cdot \Pi_{-j} < -EU'_j \cdot \Pi'_{-j}$. Let us distinguish two cases:

a) if EU_j is increasing, $\frac{\partial q^N}{\partial c} > 0$ if

$$\frac{EU_j''}{EU_j'} < -\frac{\Pi_{-j}'}{\Pi_{-j}}. \quad (20)$$

b) if EU_j is decreasing, $\frac{\partial q^N}{\partial c} > 0$ if

$$-\frac{EU_j''}{EU_j'} < \frac{\Pi_{-j}'}{\Pi_{-j}}. \quad (21)$$

Both (20) and (21) are satisfied if $\left| \frac{EU_j''}{EU_j'} \right| < \left| \frac{\Pi_{-j}'}{\Pi_{-j}} \right|$. Where “low enough” in the statement of the Lemma should be read as “lower than $\left| \frac{\Pi_{-j}'}{\Pi_{-j}} \right|$ ”. ■

Proof. Corollary 1. It is easy to see that the Arrow-Pratt coefficient is always “sufficiently high” when EU_j is concave in q^N .

The proof for the case of a convex and increasing EU_j can be easily derived from the discussion of Lemma 6 in Section 3.2.

The Corollary holds also when EU_j is convex and decreasing. In this case, a “more favorable” threshold means a lower one. Observe that $\frac{EU_j''}{EU_j'}$ is negative, whereas condition (21) for a higher q requires a low value for $-\frac{EU_j''}{EU_j'}$; thus, for a lower q , a high curvature is required: also in this case, the condition for a more favorable q is a “sufficiently high” value of the Arrow-Pratt coefficient, as in inequality (21). ■

Proof. Proposition 6. From Lemmas 1 and 3 and Proposition 5, it is easy to see that, if j is confident, any of the changes (a-c) leads EU_j to shift downwards and/or rightwards. If j is non-confident, (a-c) yield downward and/or leftward shifts in EU_j . Thus, in both cases results in Lemma 5 and Lemma 6 apply. ■

4.2 A discrete model for a small number of agents

In this Section we present the “discrete” version of the model in Section 2. Our aim is to show that the main results in that Section hold, and that in some cases we need the assumption of “sufficiently

many players”.

Consider that a discrete model can be built only for the case of unweighted votes. It is impossible to get exact probability distributions when agents have different numbers of votes. Thus we assume n agents with one vote each. For simplicity, let us normalize the status quo utility to zero. The exact probabilities of winning and losing originate from two binomial distributions with parameters (n, p) and $(n, 1 - p)$, respectively. Thus, for any majority threshold q , we can write:

$$\frac{EU_j(q)}{u_j(\alpha)} = \sum_{x=q-1}^{n-1} \binom{n-1}{x} p^x (1-p)^{n-1-x} - RASQ_j \sum_{x=q}^{n-1} \binom{n-1}{x} (1-p)^x p^{n-1-x}$$

Call $\Delta(q) \equiv \frac{q}{n-q} \left(\frac{p}{1-p} \right)^{2q-n}$. The first-order difference of $\frac{EU_j(q)}{u_j(\alpha)}$ is $RASQ_j - \Delta(q)$. It is easy to see that

$$EU_j(q+1) < EU_j(q) \quad \text{iff} \quad \Delta(q) > RASQ_j$$

and

$$EU_j(q+1) > EU_j(q) \quad \text{iff} \quad \Delta(q) < RASQ_j.$$

Observe that if $p > \frac{1}{2}$ then $\Delta(q)$ is increasing in q and it is larger than one for all q . This implies that if $RASQ_j \leq 1$, the most preferred threshold is the simple majority. This is consistent with part b) of Proposition 1.

If $RASQ_j > 1$, then EU_j is possibly not monotonic in q . Thus there might be several local maxima for EU_j ; among them j will choose the global one. Below we show that if n is sufficiently large the first-order difference of $EU_j(q)$ is positive in the simple majority and negative in unanimity. This implies that the global maximum is a supermajority. Assume that n is odd, thus the simple majority is $\frac{n+1}{2}$. Observe that $\Delta(\frac{n+1}{2}) = \frac{p}{n-1} \frac{n+1}{1-p}$. Thus, if n and $RASQ_j$ are sufficiently large then $\Delta(\frac{n+1}{2}) < RASQ_j$. Thus $EU_j(q)$ is increasing when q is the bare majority. Moreover, $\Delta(n-1) = (n-1) \left(\frac{p}{1-p} \right)^{n-2}$. For large n , $\Delta(n-1) > RASQ_j$, then EU_j decreases when q approaches unanimity. Thus the global maximum must be an interior supermajority. If instead

$\Delta(n-1) < RASQ_j$ then the maximum is possibly unanimity. This is consistent with part a) of Proposition 1.

Now consider $p < \frac{1}{2}$. We can examine the shape of $EU_j(q)$ by studying the sign of $\Delta(q) - \Delta(q+1)$. If it is positive then possibly $\Delta(q) > RASQ_j$ holds for low thresholds and $\Delta(q) < RASQ_j$ holds for high thresholds; i.e. $EU_j(q)$ is convex. Of course, this is not always the case (and again this is consistent with the continuous model). Observe, however that

$$\Delta(q) - \Delta(q+1) = \left(\frac{p}{1-p} \right)^{2q-n} \left[\frac{q}{n-q} - \frac{q+1}{n-q-1} \left(\frac{p}{1-p} \right)^2 \right].$$

In the right-hand side the first term is positive and the term in square brackets is positive if p is sufficiently low. In this case, $EU_j(q)$ is convex, provided $RASQ_j$ is sufficiently low. This is consistent with the analysis in Section 2.1.2. The proof of other results in that Section is straightforward.